Chapter
Inverse Trigonometric Functions

Topic-1: Trigonometric Functions & Their Inverses, Domain & Range of Inverse Trigonometric Functions, Principal Value of Inverse Trigonometric Functions

1 MCQs with One Correct Answer

1. Considering only the principal values of the inverse trigonometric functions, the value of

$\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is **[Adv. 2024]**

(a) $\frac{7}{24}$ (b) $\frac{-7}{24}$

(c) $\frac{-5}{24}$ (d) $\frac{5}{24}$

2 Integer Value Answer/ Non-Negative Integer

2. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$

in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ equals ______ **[Adv. 2019]**

6 MCQs with One or More than One Correct Answer

3. If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is (are) **[Adv. 2015]**

(a) $\cos\beta>0$ (b) $\sin\beta<0$

(c) $\cos(\alpha+\beta)>0$ (d) $\cos\alpha<0$

4. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is **[1986 - 2 Marks]**

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) none

7 Match the Following

5. Match List I with List II and select the correct answer using the code given below the lists : **[Adv. 2013]**

List I	List II
P. $\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$ takes value	1. $\frac{1}{2}\sqrt{\frac{5}{3}}$
Q. If $\cos x+\cos y+\cos z=0=\sin x+\sin y+\sin z$ then possible value of $\cos\frac{x-y}{2}$ is	2. $\sqrt{2}$
R. If $\cos\left(\frac{\pi}{4}-x\right)\cos 2x+\sin x\sin 2\sec x=\cos x\sin 2x\sec x+\cos\left(\frac{\pi}{4}+x\right)\cos 2x$ then possible value of $\sec x$ is	3. $\frac{1}{2}$

S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x \neq 0$, then possible value of x is — 4. 1

Codes:

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

Topic-2: Properties of Inverse Trigonometric Functions, Infinite Series of Inverse Trigonometric Functions

1 MCQs with One Correct Answer

1. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is [Adv. 2013]

(a) $\frac{23}{25}$ (b) $\frac{25}{23}$ (c) $\frac{23}{24}$ (d) $\frac{24}{23}$

2. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{1/2}=$ [2008]

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x

(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

3. The value of x for which $\sin(\cot^{-1}(1+x))=\cos(\tan^{-1}x)$ is [2004S]

(a) 1/2 (b) 1 (c) 0 (d) –1/2

4. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals [2001S]

(a) 1/2 (b) 1 (c) –1/2 (d) –1

5. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\pi/2$ is [1999 - 2 Marks]

(a) zero (b) one (c) two (d) infinite

6. If we consider only the principle values of the inverse trigonometric functions then the value of

$\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right)$ is [1994]

(a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$

7. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is [1983 - 1 Mark]

(a) $\frac{6}{17}$ (b) $\frac{7}{16}$ (c) $\frac{16}{7}$ (d) none

2 Integer Value Answer/ Non-Negative Integer

8. Let $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, for $x\in\mathbb{R}$. Then the number of real solutions of the equation $\sqrt{1+\cos(2x)}=\sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ is equal to [Adv. 2023]

9. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)=\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i-\sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is ______. (Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0,\pi]$, respectively.) [Adv. 2018]

3 Numeric/ New Stem Based Questions

10. Considering only the principal values of the inverse trigonometric functions, the value of [Adv. 2022]

$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$ is ______.

4 Fill in the Blanks

11. The greater of the two angles $A=2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B=3\sin^{-1}(1/3)+\sin^{-1}(3/5)$ is ______ [1989 - 2 Marks]

12. The numerical value of $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right\}$ is equal to ______ [1984 - 2 Marks]

13. Let a, b, c be positive real numbers. Let

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}.$$

Then $\tan\theta =$ ________ **[1981 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

14. For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the sum of all solutions of the equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$ for $0 < |y| < 3$. is equal to **[Adv. 2023]**

(a) $2\sqrt{3}-3$ (b) $3-2\sqrt{3}$ (c) $4\sqrt{3}-6$ (d) $6-4\sqrt{3}$

15. For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$ where for any $x \in \mathbb{R}$, $\cot^{-1} x \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) TRUE? **[Adv. 2021]**

(a) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(b) $\lim_{n \to \infty} \cot(S_n(x)) = x$, for all $x > 0$

(c) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(d) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

16. For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0, \pi]$, which of the following options is/are correct? **[Adv. 2019]**

(a) $\lim_{n\to\infty} f(n) = \frac{1}{2}$

(b) $f(4) = \frac{\sqrt{3}}{2}$

(c) If $\alpha = \tan\left(\cos^{-1} f(6)\right)$, then $\alpha^2 + 2\alpha - 1 = 0$

(d) $\sin(7\cos^{-1} f(5)) = 0$

7 Match the Following

17. Let (x, y) be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}.$$ **[2007]**

Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I	Column II
(A) If $a = 1$ and $b = 0$, then (x, y)	(p) lies on the circle $x^2 + y^2 = 1$
(B) If $a = 1$ and $b = 1$, they (x, y)	(q) lies on $(x^2-1)(y^2-1) = 0$
(C) If $a = 1$ and $b = 2$, then (x, y)	(r) lies on $y = x$
(D) If $a = 2$ and $b = 2$, then (x, y)	(s) lies on $(4x^2-1)(y^2-1) = 0$

18. Match the following **[2006 - 6M]**

Column I	Column II
(A) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then $\tan t =$	(p) 1
(B) Sides a, b, c of a triangle ABC are in AP and $\cos\theta_1 = \frac{a}{b+c}$, $\cos\theta_2 = \frac{b}{a+c}$, $\cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$	(q) $\frac{\sqrt{5}}{3}$
(C) A line is perpendicular to $x + 2y + 2z = 0$ and passes through (0, 1, 0). The perpendicular distance of this line from the origin is	(r) $\frac{2}{3}$

10 Subjective Problems

19. Find the value of : $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$, where $0 \le \cos^{-1}x \le \pi$ and $-\pi/2 \le \sin^{-1}x \le \pi/2$. **[1981 - 2 Marks]**

Answer Key

Topic-1 : Trigonometric Functions & Their Inverses, Domain & Range of Inverse Trigonometric Functions

1. (b) **2.** (0) **3.** (b, c, d) **4.** (d) **4.** (b)

Topic-2 : Properties of Inverse Trigonometric Functions, Infinite Series of Inverse Trigonometric Functions

1. (b) **2.** (c) **3.** (d) **4.** (b) **5.** (c) **6.** (d) **7.** (d) **8.** (3) **9.** (2)
10. (2.36) **11.** (A) **12.** (–7/17) **13.** (0) **14.** (c) **15.** (a, b) **16.** (b, c, d)
17. (A) $\rightarrow$ p; (B) $\rightarrow$ q; (C) $\rightarrow$ p; (D) $\rightarrow$ s **18.** (A) $\rightarrow$ (p); (B) $\rightarrow$ (r); (C) $\rightarrow$ (q)

Hints & Solutions

Topic-1: Trigonometric Functions & Their Inverses, Domain & Range of Inverse Trigonometric Functions, Principal Value of Inverse Trigonometric Functions

1. **(b)** We have, $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

Let $\sin^{-1}\frac{3}{5}=\alpha, 2\cos^{-1}\frac{2}{\sqrt{5}}=\beta \Rightarrow \cos\frac{\beta}{2}=\frac{2}{\sqrt{5}}$

$\because \sin\alpha=\frac{3}{5}\Rightarrow \tan\alpha=\frac{3}{4}$

$$\tan\beta=\frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}=\frac{4}{3}$$

$$\Rightarrow \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\frac{3}{4}-\frac{4}{3}}{1+1}=-\frac{7}{24}$$

2. **(0)** $\sec^{-1}\left[\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right]$

$$=\sec^{-1}\left[\frac{1}{2}\sum_{k=0}^{10}\frac{1}{2\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}+\frac{\pi}{2}\right)}\right]$$

$$=\sec^{-1}\left[\frac{1}{2}\sum_{k=0}^{10}\frac{-1}{\sin\left(\frac{7\pi}{6}+k\pi\right)}\right]$$

$$=\sec^{-1}\left[\frac{1}{2}\sum_{k=0}^{10}\frac{-1}{\sin\left((k+1)\pi+\frac{\pi}{6}\right)}\right]$$

If k is an even integer, then

$$\sin\left((k+1)\pi+\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=\frac{-1}{2}$$

If k is an odd integer, then $\sin\left((k+1)\pi+\frac{\pi}{6}\right)$

$$=\sin\frac{\pi}{6}=\frac{1}{2}$$

$$\therefore \sum_{k=0}^{9}\frac{1}{\sin\left((k+1)\pi+\frac{\pi}{6}\right)}=0$$

Hence $\sec^{-1}\left[\frac{1}{2}\sum_{k=0}^{10}\frac{-1}{\sin\left((k+1)\pi+\frac{\pi}{6}\right)}\right]$

$$=\sec^{-1}\left[\frac{1}{2}\left(\frac{-1}{-\frac{1}{2}}\right)\right]=\sec^{-1}(1)=0$$

3. **(b, c, d)** $\alpha=3\sin^{-1}\frac{6}{11}>3\sin^{-1}\frac{1}{2}=\frac{\pi}{2}\Rightarrow \alpha>\frac{\pi}{2}$

$\therefore \quad \cos\alpha<0$

$\beta=3\cos^{-1}\frac{4}{9}>3\cos^{-1}\frac{1}{2}=\pi\Rightarrow \beta>\pi$

$\therefore \quad \cos\beta<0$ and $\sin\beta<0$

Now $\alpha+\beta>\frac{3\pi}{2}, \quad \therefore \cos(\alpha+\beta)>0$

4. **(d)** The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$$=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}(\sin\pi/3)=\pi/3$$

5. **(b)** (P) $\left[\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right]^{\frac{1}{2}}$

$$=\left[\frac{1}{y^2}\left(\frac{\cos\left(\cos^{-1}\frac{1}{\sqrt{1+y^2}}\right)+y\sin\left(\sin^{-1}\frac{y}{\sqrt{1+y^2}}\right)}{\cot\left(\cot^{-1}\frac{\sqrt{1-y^2}}{y}\right)+\tan\left(\tan^{-1}\frac{y}{\sqrt{1-y^2}}\right)}\right)^2+y^4\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{y^2}\left(\frac{\frac{\sqrt{1+y^2}}{1}}{\frac{1}{y(\sqrt{1-y^2})}}\right)^2+y^4\right]^{\frac{1}{2}}$$

$$=\left(1-y^4+y^4\right)^{\frac{1}{2}}=1 \qquad \therefore \quad (P)\rightarrow(4)$$

(Q) $\cos x + \cos y = -\cos z$ ...(i)

and $\sin x + \sin y = -\sin z$ ...(ii)

On squaring (i) and (ii) and then adding, we get

$(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = \cos^2 z + \sin^2 z$

$\Rightarrow \quad 2 + 2\cos(x - y) = 1$

$\Rightarrow \quad 4\cos^2\frac{x-y}{2} = 1 \Rightarrow \cos\frac{x-y}{2} = \pm\frac{1}{2}$

$\therefore \quad Q \to (3)$

(R) $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x$

$= \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$

$\Rightarrow \cos 2x\left[\cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]$

$= \sin 2x \sec x (\cos x - \sin x)$

$\Rightarrow 2\sin\frac{\pi}{4}\sin x \cos 2x = 2\sin x(\cos x - \sin x)$

$\Rightarrow 2\sin x\left[\frac{1}{\sqrt{2}}(\cos^2 x - \sin^2 x) - (\cos x - \sin x)\right] = 0$

$\Rightarrow 2\sin x(\cos x - \sin x)\left(\frac{\cos x + \sin x}{\sqrt{2}} - 1\right) = 0$

$\Rightarrow \sin x = 0$ or $\tan x = 1$ or $\cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow x = 0$ or $\frac{\pi}{4} \Rightarrow \sec x = 1$ or $\sqrt{2}$

$\therefore (R) \to (2, 4)$

(S) $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1} x\sqrt{6}\right)$

$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{1+6x^2}} \Rightarrow x = \pm\frac{1}{2}\sqrt{\frac{5}{3}}$

$\therefore (S) \to (1)$

Hence (P) $\to$ (4), (Q) $\to$ (3), (R) $\to$ (2, 4), (S) $\to$ (1)

Topic-2: Properties of Inverse Trigonometric Functions, Infinite Series of Inverse Trigonometric Functions

1. (b) $\cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right) = \cot^{-1}[1 + n(n+1)]$

$= \tan^{-1}\left[\frac{(n+1) - n}{1 + (n+1)n}\right] = \tan^{-1}(n+1) - \tan^{-1} n$

$\therefore \sum_{n=1}^{23}[\tan^{-1}(n+1) - \tan^{-1} n] = \tan^{-1} 24 - \tan^{-1} 1 = \tan^{-1}\frac{23}{25}$

$\therefore \cot\left[\sum_{n=1}^{23}\cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right] = \cot\left[\tan^{-1}\frac{23}{25}\right] = \frac{25}{23}$

2. (c) $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1\right]^{\frac{1}{2}}$

$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) + \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)\right\}^2 - 1\right]^{\frac{1}{2}}$

$= \sqrt{1+x^2}\left[\left\{x.\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{\frac{1}{2}}$

$= \sqrt{1+x^2}\left[\left(\sqrt{1+x^2}\right)^2 - 1\right]^{\frac{1}{2}} = x\sqrt{1+x^2}$

3. (d) $\sin\left[\cot^{-1}(1+x)\right] = \cos(\tan^{-1} x)$

$\Rightarrow \sin\left[\sin^{-1}\left(\frac{1}{\sqrt{1+(1+x)^2}}\right)\right] = \cos\left[\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$

$\Rightarrow \frac{1}{\sqrt{1+(1+x)^2}} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow 1 + 1 + 2x + x^2 = 1 + x^2 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

4. (b) $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}...\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4}...\right) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4}...\right) = \frac{\pi}{2} - \sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}...\right)$

$\Rightarrow \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4}...\right) = \cos^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}...\right)$

$\Rightarrow x^2 - \frac{x^4}{2} + \frac{x^6}{4}... = x - \frac{x^2}{2} + \frac{x^3}{4}...$

On both sides we have G.P. of infinite terms.

$\therefore \frac{x^2}{1 - \left(\frac{-x^2}{2}\right)} = \frac{x}{1 - \left(-\frac{x}{2}\right)} \Rightarrow \frac{2x^2}{2 + x^2} = \frac{2x}{2 + x}$

$\Rightarrow 2x + x^3 = 2x^2 + x^3 \Rightarrow x(x-1) = 0$

$\Rightarrow x = 0, 1$ but $0 < |x| < \sqrt{2} \Rightarrow x = 1.$

5. (c) $\tan^{-1}\sqrt{x(x+1)} = \pi/2 - \sin^{-1}\sqrt{x^2 + x + 1}$

$\Rightarrow \tan^{-1}\sqrt{x(x+1)} = \cos^{-1}\sqrt{x^2 + x + 1}$

$\Rightarrow \cos^{-1}\frac{1}{\sqrt{x^2 + x + 1}} = \cos^{-1}\sqrt{x^2 + x + 1}$

$\Rightarrow x^2 + x + 1 = 1 \Rightarrow x(x+1) = 0$

$\therefore \quad x = 0, -1$ are the only two real solutions.

6. **(d)** Let $\cos^{-1}\frac{1}{5\sqrt{2}}=\alpha \Rightarrow \cos\alpha=\frac{1}{5\sqrt{2}}$

Now $\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\frac{1}{50}}=\frac{7}{5\sqrt{2}}$

$\therefore\ \tan\alpha=\frac{\sin\alpha}{\cos\alpha}=7 \Rightarrow \alpha=\tan^{-1}7$

$\Rightarrow \cos^{-1}\frac{1}{5\sqrt{2}}=\tan^{-1}7$

Also suppose $\sin^{-1}\frac{4}{\sqrt{17}}=\beta \Rightarrow \sin\beta=\frac{4}{\sqrt{17}}$

Now $\cos\beta=\sqrt{1-\sin^2\beta}=\frac{1}{\sqrt{17}}$

$\therefore\ \tan\beta=\frac{\sin\beta}{\cos\beta}=4$

$\Rightarrow\ \beta=\tan^{-1}4 \Rightarrow \sin^{-1}\frac{4}{\sqrt{17}}=\tan^{-1}4$

$\therefore\ \tan\left[\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right]$

$=\tan\,[\tan^{-1}7-\tan^{-1}4]$

$=\tan\left(\tan^{-1}\frac{7-4}{1+7\times 4}\right)$

$=\tan\left(\tan^{-1}\left(\frac{3}{29}\right)\right)=\frac{3}{29}$

7. **(d)** Let $\cos^{-1}\frac{4}{5}=\theta \Rightarrow \cos\theta=\frac{4}{5}$

Now $\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

$\therefore\ \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{3}{4}\Rightarrow\theta=\tan^{-1}\frac{3}{4}$

$\therefore\ \tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

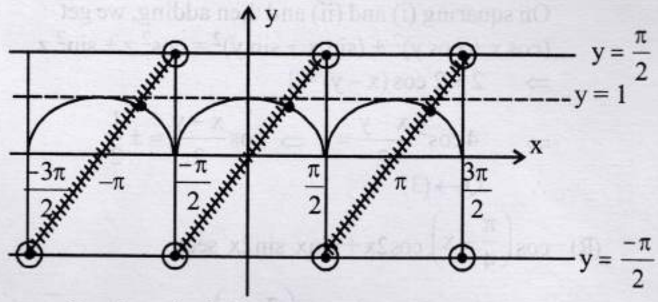
$=\tan\left[\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right]$

$=\tan\left[\tan^{-1}\left(\frac{3/4+2/3}{1-3/4\times 2/3}\right)\right]=\frac{17}{12}\times\frac{12}{6}=\frac{17}{6}$

8. **(3)** $\sqrt{1+\cos 2x}=\sqrt{2}\tan^{-1}(\tan x)$

$\Rightarrow \sqrt{2\cos^2 x}=\sqrt{2}\tan^{-1}(\tan x)$

$\Rightarrow |\cos x|=\tan^{-1}(\tan x)$



Number of solutions = Number of intersection points = 3.

9. **(2)** $\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)$

$=\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i-\sum_{i=1}^{\infty}(-x)^i\right)$

$\sin^{-1}\left(\frac{x^2}{1-x}-x\cdot\frac{\frac{x}{2}}{1-\frac{x}{2}}\right)=\sin^{-1}\left(\frac{-\frac{x}{2}}{1+\frac{x}{2}}-\frac{-x}{1+x}\right)$

[$\because$ sum of infinite terms of a G.P. $=\frac{a}{1-r}$, if $|r|<1$]

$\Rightarrow\ \frac{x^2}{1-x}-\frac{x^2}{2-x}=\frac{-x}{2+x}+\frac{x}{1+x}$

$\Rightarrow\ \frac{x^2}{1-x}-\frac{x}{1+x}+\frac{x}{2+x}-\frac{x^2}{2-x}=0$

$\Rightarrow\ \frac{x(x^2+2x-1)}{1-x^2}+\frac{x(2-3x-x^2)}{4-x^2}=0$

$\Rightarrow\ x[x^3+2x^2+5x-2]=0$

$\Rightarrow\ x=0\ \text{ or }\ x^3+2x^2+5x-2=0=p(x)\ (\text{say})$

We observe that $p(0)<0$ and $p\left(\frac{1}{2}\right)>0$

$\therefore$ One root of $p(x)=0$ lies in $\left(0,\frac{1}{2}\right)$.

Thus two solutions lie between $-\frac{1}{2}$ and $\frac{1}{2}$.

10. **(2.36)** $\left[\text{Let, }\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}=t=\tan^{-1}\frac{\pi}{\sqrt{2}}\right]$

$\left\{\text{similarly for }\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}\right\}$

Now, we have

$\frac{3}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{1}{4}\tan^{-1}\left(\frac{2\sqrt{2}\pi}{\pi^2-2}\right)+\tan^{-1}\frac{\sqrt{2}}{\pi}$

$=\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}-\frac{1}{4}\tan^{-1}\left(\frac{2\sqrt{2}\pi}{2-\pi^2}\right)$

$= \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right) - \frac{1}{4}\tan^{-1}\left(\frac{2\cdot\left(\frac{\pi}{\sqrt{2}}\right)}{1-\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$

$= \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right) - \frac{1}{4}\left(-\pi + 2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)\right)$

$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \approx 2.36$

11. $A = 2\tan^{-1}(2\sqrt{2}-1) = 2\tan^{-1}(2\times 1.414 - 1)$

$= 2\tan^{-1}(1.828) > 2\tan^{-1}\sqrt{3} = 2\pi/3$

$\therefore \quad A > 2\pi/3$ ....(i)

Now $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$

$= \sin^{-1}\left[3\times\frac{1}{3} - 4\times\frac{1}{27}\right] + \sin^{-1}(3/5)$

$= \sin^{-1}\frac{23}{27} + \sin^{-1}(0.6) =$

$\sin^{-1}(0.852) + \sin^{-1}(0.6)$

$< \sin^{-1}(\sqrt{3}/2) + \sin^{-1}(\sqrt{3}/2) = 2.\pi/3$

$\therefore \quad B < 2\pi/3$ ....(ii)

From (i) and (ii), $A > B$

12. $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left[\tan^{-1}\left(\frac{2/5}{1-(1/5)^2}\right) - \tan^{-1}(1)\right]$

$= \tan\left[\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right] = \tan\left[\tan^{-1}\left(\frac{5/12-1}{1+5/12}\right)\right]$

$= \tan(\tan^{-1}(-7/17)) = -7/17$

13. Let $a+b+c=u$, then

$\theta = \tan^{-1}\sqrt{\frac{au}{bc}} + \tan^{-1}\sqrt{\frac{bu}{ca}} + \tan^{-1}\sqrt{\frac{cu}{ab}}$

$\because \sqrt{\frac{au}{bc}} \times \sqrt{\frac{bu}{ca}} = \frac{u}{c} = \frac{a+b+c}{c} > 1$

[$\because$ a, b, c are positive real numbers]

$\therefore \quad \theta = \pi + \tan^{-1}\left[\frac{\sqrt{\frac{au}{bc}} + \sqrt{\frac{bu}{ca}}}{1 - \sqrt{\frac{au}{bc}}\sqrt{\frac{bu}{ca}}}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$

$[\because \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ *when* $xy > 1]$

$\theta = \pi + \tan^{-1}\left[\frac{\frac{a+b}{\sqrt{abc}}\sqrt{u}}{1-\frac{u}{c}}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$

$\left[\because \sqrt{\frac{au}{bc}} \times \sqrt{\frac{bu}{ca}} = \frac{u}{c}\right]$

$\theta = \pi + \tan^{-1}\left[\frac{(u-c)\sqrt{u}}{\sqrt{abc}} \times \frac{c}{-(u-c)}\right] + \tan^{-1}\sqrt{\frac{cu}{ab}}$

$\theta = \pi - \tan^{-1}\sqrt{\frac{uc}{ab}} + \tan^{-1}\sqrt{\frac{cu}{ab}} = \pi$

$[\because \tan^{-1}(-x) = -\tan^{-1}x]$

$\therefore \tan\theta = \tan\pi = 0$

14. **(c)** Case-I : $y \in (-3, 0) \Rightarrow y < 0 \Rightarrow \frac{6y}{9-y^2} < 0$

$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$

$2\tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{3}$

$y^2 - 6\sqrt{3}y - 9 = 0 \Rightarrow y = 3\sqrt{3} - 6 \ (\because y \in (-3, 0))$

Case-II : $y \in (0, 3) \Rightarrow y > 0 \Rightarrow \frac{6y}{9-y^2} > 0$

$2\tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3} \Rightarrow \sqrt{3}y^2 + 6y - 9\sqrt{3} = 0$

$y = \sqrt{3}$ or $y = -3\sqrt{3}$ (rejected)

sum $= \sqrt{3} + 3\sqrt{3} - 6 = 4\sqrt{3} - 6$

15. **(a,b)** Given that $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$

$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$

$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$

$\Rightarrow S_n(x) = \tan^{-1}(nx+x) - \tan^{-1}x$

$= \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

(a) $S_{10}(x) = \tan^{-1}\frac{10x}{1+11x^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right) (x>0)$

(Option (a) is correct)

(b) $\lim_{n\to\infty} \cot(S_n(x)) = \lim_{n\to\infty} \cot\left(\cot^{-1}\left(\frac{1+(n+1)x^2}{nx}\right)\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n} + \left(1+\frac{1}{n}\right)x^2}{x} = x \ (x>0)$

(Option (b) is correct)

(c) $S_3(x) = \tan^{-1}\dfrac{3x}{1+4x^2} = \dfrac{\pi}{4}$

$\Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$ [∵ D is negative]

(Option (c) is incorrect)

(d) For $x = 1$

$\tan(S_n(x)) = \dfrac{n}{n+2} \geq \dfrac{1}{2}$

for $n \geq 3$.

(Option (d) is incorrect)

16. **(b, c, d)** $f(n) = \dfrac{\sum_{k=0}^{n} 2\sin\left(\dfrac{k+1}{n+2}\pi\right)\sin\left(\dfrac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} 2\sin^2\left(\dfrac{k+1}{n+2}\right)\pi}$,

where n is non negative integer

$$= \frac{\sum_{k=0}^{n}\left[\cos\left(\frac{\pi}{n+2}\right) - \cos\frac{(2k+3)\pi}{n+2}\right]}{\sum_{k=0}^{n}\left[1-\cos\frac{2(k+1)\pi}{n+2}\right]}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \left[\cos\frac{3\pi}{n+2} + \cos\frac{5\pi}{n+2} + \dots\dots + \cos\frac{(2n+3)\pi}{n+2}\right]}{n+1-\left[\cos\frac{2\pi}{n+2} + \cos\frac{4\pi}{n+2} + \dots\dots + \cos\frac{2(n+1)\pi}{n+2}\right]}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \frac{\sin\frac{(n+1)\pi}{n+2}}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\frac{(2n+6)\pi}{2(n+2)}}{n+1-\frac{\sin\frac{(n+1)\pi}{n+2}}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\frac{(2n+4)\pi}{2(n+2)}}$$

$$= \frac{(n+1)\cos\frac{\pi}{n+2} + \cos\frac{\pi}{n+2}}{n+1+1} = \frac{(n+2)\cos\left(\frac{\pi}{n+2}\right)}{n+2}$$

$\therefore f(n) = \cos\left(\dfrac{\pi}{n+2}\right)$

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \cos\left(\dfrac{\pi}{n+2}\right) = 1$

∴ Option (a) is incorrect.

$f(4) = \cos\left(\dfrac{\pi}{4+2}\right) = \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

∴ Option (b) is correct

If $\alpha = \tan\left(\cos^{-1} f(6)\right)$

$= \tan\left(\cos^{-1}\left(\cos\dfrac{\pi}{8}\right)\right) = \tan\dfrac{\pi}{8}$

Now, $\tan\dfrac{\pi}{4} = 1 \Rightarrow \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\pi/8} = 1$

$\Rightarrow \dfrac{2\alpha}{1-\alpha^2} = 1 \Rightarrow \alpha^2 + 2\alpha - 1 = 0$

∴ Option (c) is correct

$\sin\left(7\cos^{-1} f(5)\right) = \sin\left(7\cos^{-1}\left(\cos\dfrac{\pi}{7}\right)\right) = \sin\left(7\times\dfrac{\pi}{7}\right)$

$= \sin\pi = 0$

∴ Option (d) is correct.

17. **(A) → p; (B) → q; (C) → p; (D) → s**

$\sin^{-1}(ax) + \cos^{-1}y + \cos^{-1}(bxy) = \dfrac{\pi}{2}$

$\Rightarrow \cos^{-1}y + \cos^{-1}(bxy) = \dfrac{\pi}{2} - \sin^{-1}(ax) = \cos^{-1}(ax)$

Let $\cos^{-1}y = \alpha$, $\cos^{-1}(bxy) = \beta$, $\cos^{-1}(ax) = \gamma$

then $y = \cos\alpha$, $bxy = \cos\beta$, $ax = \cos\gamma$

∴ We get $\alpha + \beta = \gamma$ and $\cos\beta = bxy$

$\Rightarrow \cos(\gamma - \alpha) = \cos\beta = bxy$

$\Rightarrow \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = bxy$

$\Rightarrow axy + \sin\gamma\sin\alpha = bxy \Rightarrow (a-b)xy = -\sin\alpha\sin\gamma$

$\Rightarrow (a-b)^2x^2y^2 = \sin^2\alpha\sin^2\gamma$

$= (1-\cos^2\alpha)(1-\cos^2\gamma)$

$\Rightarrow (a-b)^2x^2y^2 = (1-y^2)(1-a^2x^2)$ ....(i)

(A) For $a = 1, b = 0$, equation (i) reduces to $x^2y^2 = (1-x^2)(1-y^2) \Rightarrow x^2 + y^2 = 1$

(B) For $a = 1, b = 1$ equation (i) becomes $(1-x^2)(1-y^2) = 0 \Rightarrow (x^2-1)(y^2-1) = 0$

(C) For $a = 1, b = 2$ equation (i) reduces to $x^2y^2 = (1-x^2)(1-y^2) \Rightarrow x^2 + y^2 = 1$

(D) For $a = 2, b = 2$ equation (i) reduces to $0 = (1-4x^2)(1-y^2) \Rightarrow (4x^2-1)(y^2-1) = 0$

18. **(A) → (p); (B) → (r); (C) → (q)**

(A) $t = \sum_{i=1}^{\infty}\tan^{-1}\left(\dfrac{1}{2i^2}\right) = \sum_{i=1}^{\infty}\tan^{-1}\left[\dfrac{(2i+1)-(2i-1)}{1+4i^2-1}\right]$

$= \sum_{i=1}^{\infty}[\tan^{-1}(2i+1) - \tan^{-1}(2i-1)]$

$= \tan^{-1}3 - \tan^{-1}1 + \tan^{-1}5 - \tan^{-1}3 + \dots\dots + \tan^{-1}(2n+1) - \tan^{-1}(2n-1) + \dots\dots\ \infty$

$= \lim_{n\to\infty}[\tan^{-1}(2n+1) - \tan^{-1}1]$

$= \lim_{n\to\infty}\tan^{-1}\left[\dfrac{2n}{1+(2n+1)}\right] = \lim_{n\to\infty}\tan^{-1}\left[\dfrac{1}{1+1/n}\right]$

$= \tan^{-1}(1) = \dfrac{\pi}{4} \Rightarrow \tan t = 1$, (A) → (p)

(B) $\because a, b, c$ are in AP $\Rightarrow 2b = a + c$

Now $\cos\theta_1 = \dfrac{a}{b+c} \Rightarrow \dfrac{1-\tan^2\theta_1/2}{1+\tan^2\theta_1/2} = \dfrac{a}{b+c}$

$\Rightarrow \tan^2\dfrac{\theta_1}{2} = \dfrac{b+c-a}{b+c+a}$

Similarly, $\tan^2\dfrac{\theta_3}{2} = \dfrac{a+b-c}{a+b+c}$

$\Rightarrow \tan^2\dfrac{\theta_1}{2} + \tan^2\dfrac{\theta_3}{2} = \dfrac{2b}{a+b+c} = \dfrac{2b}{3b} = \dfrac{2}{3}$, (B)$\rightarrow$(r)

(C) Equation of line through (0, 1, 0) and perpendicular to

$x + 2y + 2z = 0$ is $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z}{2} = \lambda$

For some value of λ, the foot of perpendicular from origin to line is $(\lambda, 2\lambda+1, 2\lambda)$

Direction ratios of this $\perp$ from origin are $\lambda, 2\lambda+1, 2\lambda$

$\therefore 1\lambda + 2(2\lambda+1) + 2.2\lambda = 0 \Rightarrow \lambda = -\dfrac{2}{9}$

$\therefore$ Foot of perpendicular is $\left(-\dfrac{2}{9}, \dfrac{5}{9}, -\dfrac{4}{9}\right)$

Hence required distance

$= \sqrt{\dfrac{4}{81} + \dfrac{25}{81} + \dfrac{16}{81}} = \sqrt{\dfrac{45}{81}} = \dfrac{\sqrt{5}}{3}$ (C)$\rightarrow$(q)

19. $\cos(2\cos^{-1}x + \sin^{-1}x)$

$= \cos(\cos^{-1}x + \cos^{-1}x + \sin^{-1}x)$

$= \cos(\cos^{-1}x + \pi/2) \quad \{\because \cos^{-1}x + \sin^{-1}x = \pi/2\}$

$= -\sin(\cos^{-1}x) = -\sqrt{1-\cos^2(\cos^{-1}x)}$

$= -\sqrt{1-[\cos(\cos^{-1}x)]^2} = -\sqrt{1-x^2}$

At $x = \dfrac{1}{5}, \cos(2\cos^{-1}x + \sin^{-1}x) = -\sqrt{1-x^2}$

$= -\sqrt{1-1/25} = \dfrac{-2\sqrt{6}}{5}$